Rotating reference frames

In the classical approach, the inertial frame remains the true reference for the laws of mechanics. When using a rotating reference frame, the laws of physics are mapped from the most convenient inertial frame to that rotating frame. Assuming a constant rotation speed, this is achieved by adding to every object two *coordinate accelerations* which correct for the rotation of the coordinate axes.

$$\begin{aligned} \mathbf{a}_{\text{rot}} &= \mathbf{a} - 2\omega \times \mathbf{v} - \omega \times (\omega \times \mathbf{r}) \\ &= \mathbf{a} + \mathbf{a}_{\text{Coriolis}} + \mathbf{a}_{\text{centrifugal}} \end{aligned}$$

where $\mathbf{a}_{\text{rot}}$ is the acceleration relative to the rotating frame, $\mathbf{a}$ is the acceleration relative to the inertial frame, ω is the angular velocity vector describing the rotation of the reference frame, $\mathbf{v}$ is the velocity of the body relative to the rotating frame, and $\mathbf{r}$ is a vector from an arbitrary point on the rotation axis to the body. A derivation can be found in the article fictitious force.

The last term is the centrifugal acceleration, so we have:

$$\mathbf{a}_{\text{centrifugal}} = -\omega \times (\omega \times \mathbf{r}) = \omega^2 \mathbf{r}_\perp$$

where $\mathbf{r}_\perp$ is the component of $\mathbf{r}$ perpendicular to the axis of rotation.

Derivation

If we have two frames, one inertial and one rotating with a constant angular velocity $\vec{\omega}$, a time derivative of a vector in the rotating frame, $\left(\frac{d}{dt}\right)_r$, is transformed to the time derivative in the inertial frame, $\left(\frac{d}{dt}\right)_i$, by the following relation:

$$\left(\frac{d}{dt}\right)_i = \left(\frac{d}{dt}\right)_r + \vec{\omega}\times$$

This relationship is one between two operators. Now, acceleration is the second derivative of position with respect to time. So, applying the above transformation to the position vector $\vec{r}$ once gets you:

$$\dot{\vec{r}}_i = \left(\frac{d\vec{r}}{dt}\right)_i = \left(\frac{d\vec{r}}{dt}\right)_r + \omega \times \vec{r}$$

Putting $\dot{\vec{r}}_i$ back into the transformation, you get:

$$\ddot{\vec{r}}_i = \left(\frac{d\dot{\vec{r}}}{dt}\right)_i = \left(\frac{d\dot{\vec{r}}}{dt}\right)_r + \omega \times \dot{\vec{r}}$$

$$\ddot{\vec{r}}_i = \left(\frac{d^2\vec{r}}{dt^2}\right)_i = \left(\frac{d}{dt}\right)_r \left(\left(\frac{d\vec{r}}{dt}\right)_r + \omega \times \vec{r}\right) + \vec{\omega} \times \left(\left(\frac{d\vec{r}}{dt}\right)_r + \omega \times \vec{r}\right)$$

Because $\vec{\omega}$ is a contant vector - that is the rotating reference frame is rotating constantly in the same direction - its time derivative is zero. So, simplifying:

$$\ddot{\vec{r}}_i = \left(\frac{d^2\vec{r}}{dt^2}\right)_i = \left(\frac{d^2\vec{r}}{dt^2}\right)_r + \omega \times \left(\frac{d\vec{r}}{dt}\right)_r + \vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_r + \omega \times \omega \times \vec{r}$$

$$\ddot{\vec{r}}_i = \left(\frac{d^2\vec{r}}{dt^2}\right)_i = \left(\frac{d^2\vec{r}}{dt^2}\right)_r + 2\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_r + \omega \times \omega \times \vec{r}$$

Finally, putting in $\vec{a}$ for $\left(\frac{d^2\vec{r}}{dt^2}\right)$ and $\vec{v}_r$ for $\left(\frac{d\vec{r}}{dt}\right)_r$, we get the following:

$$\vec{a}_i = \vec{a}_r + 2\vec{\omega} \times \vec{v}_r + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

Moving things to the other side, but reversing one cross-product in each term, you find:

$$\vec{a}_r = \vec{a}_i + 2\vec{v}_r \times \vec{\omega} + \vec{\omega} \times (\vec{r} \times \vec{\omega})$$

This tells us that $\vec{a}_r$, the acceleration of some object at $\vec{r}$ as observed by someone at rest in the rotating frame is equal to the acceleration, $\vec{a}_i$, as observed by an observer in the inertial, non-rotating frame, plus $2\vec{v}_r \times \vec{\omega}$, which is the coriolis effect's contribution to the acceleration, and $\vec{\omega} \times (\vec{r} \times \vec{\omega})$, which is the centrifugal acceleration term.

Centrifugal force (from Latin *centrum* "center" and *fugere* "to flee") is a term which may refer to two *different* forces which are related to rotation. Both of them are oriented away from the axis of rotation, but the object on which they are exerted differs.

- The **reactive** centrifugal force is the reaction to the centripetal force. This is equal in magnitude to the centripetal force, directed away from the center of rotation, and is exerted by the rotating object upon the object which exerts the centripetal force. As it is an actual force, it is always present, independent of the choice of reference frame.

- The **fictitious** centrifugal force appears when a rotating reference frame is used for analyzing the system. The centrifugal force is exerted on all objects, and directed away from the axis of rotation.

Both of the above can be observed in action on a passenger riding in a car. If the car swerves around a corner, the passenger's body pushes against the outer edge of the car. This is the reactive centrifugal force, which is called a reaction force because it results from passive interaction with the car which actively pushes against the body.

Using a reference frame which is fixed relative to the car (a model which those inside the car will often find natural) and while ignoring its rotation, it looks like an external force is pulling the passenger out of the car. This is the fictitious centrifugal force, so called because it is not an actual force exerted by some other object.